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BOSTON UNIVERSITY

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Thesis

FUNDAMENTAL PROPERTIES OF DETERMINANTS

AND SOME OF THEIR APPLICATIONS

Submitted by

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(A. B. Northwest Nazarene College, 1928)

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and Some of Their Applications

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FUNDAMENTAL PROPERTIES OF DETERMINANTS

AND SOME OF THEIR APPLICATIONS

I Determinants in General

1. Definition and Notation

A square array of n^2 numbers, called elements, arranged in n rows and n columns of n numbers each, and inclosed by two vertical bars is called a determinant of the n th order. It is written thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 & \dots\dots\dots & r_1 \\ a_2 & b_2 & c_2 & \dots\dots\dots & r_2 \\ a_3 & b_3 & c_3 & \dots\dots\dots & r_3 \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ a_n & b_n & c_n & \dots\dots\dots & r_n \end{vmatrix}$$

It is a symbol for the sum of all the products that can be made (1) by taking as factors one and only one element from each row and each column, and (2) by giving to such a product a plus or a minus sign according as the number of inversions in the subscripts is even or odd, after the letters of the product are written in the order in which they appear in the first row of the determinant.

The sum of these signed products is called the expansion of the determinant.

THE DETERMINANTS OF THE

AND SOME OF THEIR APPLICATIONS

1. Determinants in General

1.1. Definition and Notation

A square array of n^2 numbers, called elements, arranged in n rows and n columns of a square array, and enclosed by a vertical bar is called a determinant of the n th order. It is written thus:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

It is a symbol for the sum of all the products that can be obtained by taking one element from each row and one from each column, and by giving to each element a sign $+$ or $-$ according to the number of interchanges in the arrangement of the letters of the product and the letters of the first row of the determinant.

The sum of these signed products is called the expansion of the determinant.

The principal diagonal of the determinant is the line from the upper left-to the lower right-hand corner. In the above illustration the principal diagonal is the product a, b_2, c_3, \dots, r_n .

The secondary diagonal is the line from the lower left-to the upper right-hand corner.

From the above definition, a third order determinant would be written,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and from our definition of expansion, it may be expanded as follows:

$$D = a_1 b_2 c_3 + a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Other methods of notation are used, one of the most common being that of the principal diagonal; i. e., the determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

would be written (a, b_2) and (a, b_2, c_3) respectively. Another method of notation is to write simply (a, b) and (a, b, c)

2. Some Important Properties of Determinants

Property 1. The value of a determinant is not changed if corresponding rows and columns are interchanged.

$$\text{Illustration: } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property 2. If all elements of a column, or row, of a determinant are multiplied by the same number, k , the value of the determinant is multiplied by k .

$$\text{Illustration: } k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} ka_1 & kb_1 \\ ka_2 & kb_2 \end{vmatrix}$$

Property 3. If two columns, or rows, are interchanged, the sign of the determinant is changed.

$$\text{Illustration: } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$$

Property 4. If two columns, or rows, are identical, the value of the determinant is zero.

$$\text{Illustration: } \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0$$

This leads to the corollary: If two columns, or rows, are equimultiples or differ only in signs, the determinant is equal to zero.

2. If two columns of a determinant

are identical, the value of the determinant is zero.

Illustration: The value of the determinant

$$\begin{vmatrix} a_1 & a_1 & a_1 \\ b_1 & b_1 & b_1 \\ c_1 & c_1 & c_1 \end{vmatrix} = 0$$

Property 3. If all elements of a column, or row, of a

determinant are multiplied by the same number, the value

of the determinant is multiplied by it.

$$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ kb_1 & kb_2 & kb_3 \\ kc_1 & kc_2 & kc_3 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property 4. If two columns, or rows, are interchanged,

the sign of the determinant is changed.

$$\begin{vmatrix} a_1 & a_2 \\ a_2 & a_1 \end{vmatrix} = - \begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix}$$

Property 5. If two columns, or rows, are identical, the

value of the determinant is zero.

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \\ a_3 & a_3 \end{vmatrix} = 0$$

This leads to the corollary: If two columns, or rows,

of a determinant are identical, the determinant is zero.

to zero.

Property 5. If all elements of a column, or row, are zero, the value of the determinant is zero.

$$\text{Illustration: } \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0.$$

Property 6. If each element of some column, or row, is a binomial, the determinant can be expressed as the sum of two determinants; the first of which is obtained from the given determinant by substituting for the binomial elements the first terms of the binomials, and the second determinant is obtained from the given determinant by substituting for the binomial elements the second terms of the binomials.

$$\text{Illustration: } \begin{vmatrix} a_1 & (b_1 + b'_1) & c_1 \\ a_2 & (b_2 + b'_2) & c_2 \\ a_3 & (b_3 + b'_3) & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b'_1 & c_1 \\ a_2 & b'_2 & c_2 \\ a_3 & b'_3 & c_3 \end{vmatrix}.$$

Similarly, if each element of some column, or row, is the sum of n numbers, the determinant may be expressed as the sum of n determinants.

Property 7. The value of a determinant is not changed if to each element of any column, or row, is added k times the corresponding element of some other column, or row.

$$\text{Illustration: } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & (b_1 + k c_1) & c_1 \\ a_2 & (b_2 + k c_2) & c_2 \\ a_3 & (b_3 + k c_3) & c_3 \end{vmatrix}.$$

3. Evaluation of Determinants

(1) A convenient method of evaluating a third order de-

Property 3. If all elements of a column, or row, are

zero, the value of the determinant is zero.

$$\text{Illustration: } \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$$

Property 4. If each element of some column, or row, is a binomial, the determinant can be expressed as the sum of two determinants, the first of which is obtained from the given determinant by substituting for the binomial element the first term of the binomial, and the second determinant is obtained from the given determinant by substituting for the binomial element the second term of the binomial.

$$\text{Illustration: } \begin{vmatrix} a_1 & (b_1 + c_1) & d_1 \\ a_2 & (b_2 + c_2) & d_2 \\ a_3 & (b_3 + c_3) & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$$

Similarly, if each element of some column, or row, is the sum of n binomials, the determinant may be expressed as the sum of n determinants.

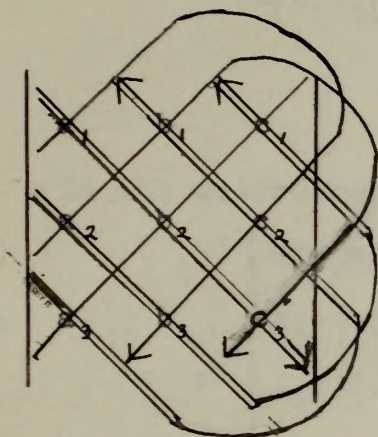
Property 5. The value of a determinant is not changed if to each element of any column, or row, is added k times the corresponding element of some other column, or row.

$$\text{Illustration: } \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = \begin{vmatrix} a & b & c \\ a + k(a) & b + k(b) & c + k(c) \\ a + k(a) & b + k(b) & c + k(c) \end{vmatrix}$$

5. Expansion of Determinants

(1) A determinant of order n is evaluated by adding a zero

terminant is indicated by the adjoining diagram and is described as follows:



The terms composed of elements of the principal diagonal and minor diagonals parallel to it are positive, (represented by double lines), and terms composed of elements of the secondary diagonal and minor diagonals parallel to it are negative, (represented by single lines)

(2) Evaluation by minors.

Definition: In any determinant, if the row and column containing any given element, a , are blotted out, the determinant formed from the remaining elements is called the minor of a .

Illustration: In $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

To evaluate a determinant by minors according to the elements of a given column, or row: (a) multiply each element of the column, or row, by its minor, and give the product a plus or a minus sign according as the sum of the numbers of the row and column containing the element is even or odd. (b) Take the sum of these signed products. This sum is the value of the given determinant.

termant is indicated by the following diagram and is as-
cribed as follows:



The terms composed of elements of
the principal diagonal and minor
diagonals parallel to it are posi-
tive, (represented by double lines),
and terms composed of elements of
the secondary diagonal and minor
diagonals parallel to it are nega-
tive, (represented by single lines).

(2) Evaluation by minors.

Definition: In any determinant, if the row and column
containing any given element, a_{ij} , are deleted, the deter-
minant formed from the remaining elements is called the

minor of a_{ij} .

Illustration: In
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
 the minor of a_{12} is
$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

To evaluate a determinant by minors according to the rule
delete a row and column, or row and column, or row and column
of the matrix, or row, or column, and give the product of
the row and column containing the element is even or odd.
(b) Take the sum of these signed products. This sum is the
value of the given determinant.

Illustration:
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Important Property

In the determinant

$$D = \begin{vmatrix} a_1' & a_1'' & \dots & a_1^m \\ a_2' & a_2'' & \dots & a_2^m \\ \dots & \dots & \dots & \dots \\ a_m' & a_m'' & \dots & a_m^m \end{vmatrix}$$

if A_h^s is the minor of a_h^s , then the function

$$a_1^p A_1^s + a_2^p A_2^s + \dots + a_m^p A_m^s$$

is equal to D when P equals S and is equal to zero when P does not equal S .

It is evident that when $P = S$, the above function is simply the expansion of D by minors and hence equals D .

An illustration will show that when $P \neq S$, the terms cancel each other and the above function equals zero. Consider the determinant

$$D = \begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix}$$

Let $P = 11$ and $S = 111$. Then the function becomes

$$a_1''(a_2' a_3'' - a_3' a_2'') - a_1'''(a_2' a_3''' - a_3' a_2''') + a_1'(a_2'' a_3''' - a_3'' a_2''') = 0.$$

Similarly, this function vanishes for a determinant of any order.

(3) It is often convenient to use the following method to simplify the evaluation of a determinant: (a) By use of property 7 reduce all but one of the elements of a certain column, or row, to zero. (b) Then expand the resulting determinant by minors according to the elements of the column, or row, containing the zeros. If desirable this minors may in turn be evaluated by this method.

(4) Laplace's Development.

Definition: A system of m n quantities arranged in a rectangular array of m rows and n columns is called a matrix.

Consider each element of a determinant of the n th order a one-rowed minor. We then have a method of pairing off each one-rowed minor of the given determinant with one of its $(n-1)$ -rowed minors. Similarly, if M is a two-rowed minor of a determinant of the n th order D , we may pair it off against the $(n-2)$ -rowed minor N obtained by striking out from D the two rows and columns which are represented in M . These two minors M and N are said to be complementary. Hence in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix}$$

the two minors

$$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

and

$$\begin{vmatrix} b_1 & d_1 & e_1 \\ b_4 & d_4 & e_4 \\ b_5 & d_5 & e_5 \end{vmatrix}$$

are complementary.

In general, we have the

Definition: If D is a determinant of the n th order and M one of its k -rowed minors, then $(n-k)$ -rowed minor N , obtained by striking out from D all the rows and columns represented in M , is called the complement of M . Conversely, M is the complement of N .

Definition: If M is the m -rowed minor of D in which the rows k_1, \dots, k_m and the columns l_1, \dots, l_m are represented, then the algebraic complement of M is defined by the equation

$$\text{Alg. Com. of } M = (-1)^{k_1 + \dots + k_m + l_1 + \dots + l_m} (\text{Comp. of } M).$$

Just as the elements of any column or row and their corresponding minors may be used to develop a determinant, so the k -rowed minors formed from any k rows or columns may be used with their algebraic complements to obtain a more general method of development. This method is known as Laplace's Development and may be stated as follows: Pick out any m rows or columns from a determinant D , and form all the m -rowed determinants from this matrix. The sum of the products of each of these minors by its algebraic complement is the value of D .

To develop by this method the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

choose the first two columns and expand. Then

$$\begin{aligned} (a_1, b_2, c, d_4) &= (a_1, b_2) (c_3, d_4) - (a_1, b_3) (c_2, d_4) + (a_1, b_4) (c_2, d_3) \\ &\quad + (a_2, b_3) (c_1, d_4) - (a_2, b_4) (c_1, d_3) + (a_3, b_4) (c_1, d_2). \end{aligned}$$

In general, we have

Definition: If A is a determinant of the n th order and A_{ij}

is the i -rowed minor, then A_{ij} is called the i -rowed minor of A .

By writing out from A all the rows and columns containing

A_{ij} , we obtain the determinant of A_{ij} . Conversely, if A is the

determinant of A_{ij} .

Definition: If A is the i -rowed minor of A in which the

rows A_1, \dots, A_{i-1} and the columns A_1, \dots, A_{i-1} are removed,

then the algebraic complement of A_{ij} is defined by the equation

$$A_{ij} = (-1)^{i+j} A_{ij}$$

that is the elements of any column or row and their

respective minors may be used to develop a determinant, so

that minors formed from any i rows or columns may be used

with their algebraic complements to obtain a more general

method of development. This method is known as Laplace's

Development and may be stated as follows: Pick out any i rows

or columns from a determinant A , and form all the i -rowed

minors from this matrix. The sum of the products of each

of these minors by its algebraic complement is the value of A .

To develop by this method the determinant

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{vmatrix}$$

choose the first two columns and expand. Then

$$\begin{aligned} (a, b, c, d) &= (a, b, c, d) - (a, b, c, d) + (a, b, c, d) \\ &+ (a, b, c, d) - (a, b, c, d) + (a, b, c, d) \end{aligned}$$

4. Multiplication of Determinants

The product of two determinants of the n th order may be expressed as a determinant of the n th order in which the element which lies in the i th row and the j th column is obtained by multiplying each element of the i th row of the first factor by the corresponding element of the j th column of the second factor and adding the results.

Illustration 1.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 A_1 + b_1 A_2 + c_1 A_3 & a_1 B_1 + b_1 B_2 + c_1 B_3 & a_1 C_1 + b_1 C_2 + c_1 C_3 \\ a_2 A_1 + b_2 A_2 + c_2 A_3 & a_2 B_1 + b_2 B_2 + c_2 B_3 & a_2 C_1 + b_2 C_2 + c_2 C_3 \\ a_3 A_1 + b_3 A_2 + c_3 A_3 & a_3 B_1 + b_3 B_2 + c_3 B_3 & a_3 C_1 + b_3 C_2 + c_3 C_3 \end{vmatrix}.$$

Illustration 2.

$$\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = \begin{vmatrix} 2+2 & 8+5 \\ 3+4 & 12+10 \end{vmatrix} = \begin{vmatrix} 4 & 13 \\ 7 & 22 \end{vmatrix} = 88 - 91 = -3.$$

5. Differentiation of Determinants

The total differential of a determinant of the n th order is the sum of n determinants, each of which is obtained from the given determinant, Δ , by substituting the differentials of the elements of a row for the elements themselves.

$$\text{Let } \Delta = (x_1, y_1, z_1, \dots, t_m).$$

Developing in terms of the elements of the i th row,

$$\Delta = x_i \bar{x}_i + y_i \bar{y}_i + z_i \bar{z}_i + \dots + t_i \bar{t}_i$$

$$\therefore d_i \Delta = dx_i \bar{x}_i + dy_i \bar{y}_i + dz_i \bar{z}_i + \dots + dt_i \bar{t}_i.$$

1. Differentiation of Determinants

The product of two determinants of the same order may be expressed as a determinant of the same order in which the elements of the first determinant are placed in the first row and the elements of the second determinant are placed in the second row, and adding the results.

Illustration 1.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} = \begin{vmatrix} a_1d_1 + b_1e_1 + c_1f_1 & a_1d_2 + b_1e_2 + c_1f_2 & a_1d_3 + b_1e_3 + c_1f_3 \\ a_2d_1 + b_2e_1 + c_2f_1 & a_2d_2 + b_2e_2 + c_2f_2 & a_2d_3 + b_2e_3 + c_2f_3 \\ a_3d_1 + b_3e_1 + c_3f_1 & a_3d_2 + b_3e_2 + c_3f_2 & a_3d_3 + b_3e_3 + c_3f_3 \end{vmatrix}$$

Illustration 2.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 & 1 \cdot 3 + 2 \cdot 6 + 3 \cdot 9 \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 & 4 \cdot 2 + 5 \cdot 5 + 6 \cdot 8 & 4 \cdot 3 + 5 \cdot 6 + 6 \cdot 9 \\ 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 & 7 \cdot 2 + 8 \cdot 5 + 9 \cdot 8 & 7 \cdot 3 + 8 \cdot 6 + 9 \cdot 9 \end{vmatrix} = \begin{vmatrix} 14 & 32 & 50 \\ 32 & 77 & 110 \\ 50 & 110 & 182 \end{vmatrix}$$

2. Differentiation of Determinants

The total differential of a determinant of the n th order is the sum of n determinants, each of which is obtained from the given determinant, Δ , by substituting the differential of the elements of a row for the elements themselves.

$$\text{Let } \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Developing in terms of the elements of row 1 we get,

$$\Delta = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} + y_1 \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1 \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

$$\Delta = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} + y_1 \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1 \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

For the total differential there must be n such expressions, each of which may be obtained from Δ by replacing the elements of some one of the rows by their differentials. Hence we have the following value of the total differential.

$$d\Delta = \begin{vmatrix} dx & dy & dz & \dots & dt \\ x_1 & y_1 & z_1 & \dots & t_1 \\ \dots & \dots & \dots & \dots & \dots \\ x_m & y_m & z_m & \dots & t_m \end{vmatrix} + \begin{vmatrix} x & y & z & \dots & t \\ dx_1 & dy_1 & dz_1 & \dots & dt_1 \\ \dots & \dots & \dots & \dots & \dots \\ x_m & y_m & z_m & \dots & t_m \end{vmatrix} + \dots + \begin{vmatrix} x & y & z & \dots & t \\ x_1 & y_1 & z_1 & \dots & t_1 \\ \dots & \dots & \dots & \dots & \dots \\ dx_m & dy_m & dz_m & \dots & dt_m \end{vmatrix}$$

If the elements of Δ are all functions of the same variable, x , $\frac{d\Delta}{dx}$ equals the sum of n determinants, each of which is obtained from Δ by substituting the derivatives of the elements of a row for the elements themselves.

$$\text{If } \Delta = \begin{vmatrix} f_1(x) & \phi_1(x) & \dots & \psi_1(x) \\ f_2(x) & \phi_2(x) & \dots & \psi_2(x) \\ \dots & \dots & \dots & \dots \\ f_m(x) & \phi_m(x) & \dots & \psi_m(x) \end{vmatrix},$$

$$\frac{d\Delta}{dx} = \begin{vmatrix} f_1'(x) & \phi_1'(x) & \dots & \psi_1'(x) \\ f_2(x) & \phi_2(x) & \dots & \psi_2(x) \\ \dots & \dots & \dots & \dots \\ f_m(x) & \phi_m(x) & \dots & \psi_m(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & \phi_1'(x) & \dots & \psi_1'(x) \\ f_1'(x) & \phi_2'(x) & \dots & \psi_2'(x) \\ \dots & \dots & \dots & \dots \\ f_m(x) & \phi_m(x) & \dots & \psi_m(x) \end{vmatrix} + \dots + \begin{vmatrix} f_1(x) & \phi_1(x) & \dots & \psi_1'(x) \\ f_2(x) & \phi_2(x) & \dots & \psi_2'(x) \\ \dots & \dots & \dots & \dots \\ f_m'(x) & \phi_m'(x) & \dots & \psi_m'(x) \end{vmatrix}$$

Illustration:

$$\Delta = \begin{vmatrix} 3x^2 & 2x \\ x & x^2 \end{vmatrix}$$

for the total differential there must be a term expression,
each of which may be obtained from Δ by replacing the element
of which one of the rows by their differentials. Hence we have
the following value of the total differential.

$$\begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{m-1}}{\partial x_1} & \frac{\partial g_{m-1}}{\partial x_2} & \dots & \frac{\partial g_{m-1}}{\partial x_n} \end{vmatrix}$$

If the elements of Δ are all functions of the same variable,
say, x , $\frac{\partial \Delta}{\partial x}$ equals the sum of a determinant, each of which
obtained from Δ by substituting the derivatives of the elements
of a row for the elements themselves.

$$\frac{\partial \Delta}{\partial x} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial x} & \dots & \frac{\partial f}{\partial x} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial x} & \dots & \frac{\partial g_1}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{m-1}}{\partial x} & \frac{\partial g_{m-1}}{\partial x} & \dots & \frac{\partial g_{m-1}}{\partial x} \end{vmatrix}$$

$$\frac{\partial \Delta}{\partial x} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial x} & \dots & \frac{\partial f}{\partial x} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial x} & \dots & \frac{\partial g_1}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{m-1}}{\partial x} & \frac{\partial g_{m-1}}{\partial x} & \dots & \frac{\partial g_{m-1}}{\partial x} \end{vmatrix}$$

$$\frac{\partial \Delta}{\partial x} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial x} & \dots & \frac{\partial f}{\partial x} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial x} & \dots & \frac{\partial g_1}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{m-1}}{\partial x} & \frac{\partial g_{m-1}}{\partial x} & \dots & \frac{\partial g_{m-1}}{\partial x} \end{vmatrix}$$

$$\frac{dA}{dx} = \begin{vmatrix} 6X & 2 \\ X & X^2 \end{vmatrix} + \begin{vmatrix} 3X^2 & 2X \\ 1 & 2X \end{vmatrix} = 6X^3 - 2X + 6X^3 - 2X = 12X^3 - 4X.$$

II Applications of Determinants

1. Applications to Theory of Equations

Solution of Linear Equations

The use of determinants in the solution of systems of simultaneous linear equations is the simplest and most common of their many applications. Solving the equations

$$\begin{cases} a_1 x + b_1 y = k_1 \\ a_2 x + b_2 y = k_2 \end{cases}$$

by ordinary algebraic methods, gives:

$$x = \frac{k_1 b_2 - k_2 b_1}{a_1 b_2 - a_2 b_1}, \quad y = \frac{a_1 k_2 - a_2 k_1}{a_1 b_2 - a_2 b_1}.$$

Writing these values of X and Y in determinant notation,

$$X = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad Y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Thus, we see that the fractions giving the values of X and Y have a common denominator which is a determinant whose elements are the coefficients of the unknowns arranged in the

same order as in the given equations. The numerator of the fraction giving the value of X is formed from the denominator by replacing each coefficient of X by the corresponding absolute term. Similarly for Y.

This method may be generalized to apply to all complete linear systems. Let the system be written:

[illegible]

The determinant of the coefficients of these equations is

$$D = \begin{vmatrix} a'_1 & a''_1 & \dots & a'_m \\ a'_2 & a''_2 & \dots & a'_m \\ \dots & \dots & \dots & \dots \\ a'_m & a''_m & \dots & a'_m \end{vmatrix}$$

Now let A_{p^s} be the minor of a_{p^s} in the above determinant, D.

Then from the property given in the section on minors, Part I, Chapter 3, the function

$$a_1^P A_1^S + a_2^P A_2^S + \dots + a_m^P A_m^S$$

is equal to D when $p = s$, and equal to zero when p and s are different superscripts. Multiplying the given equations by $A_1^s, A_2^s, \dots, A_m^s$ respectively, the sum of the resulting equations is a linear equation in which the coefficients of X^s

is equal to D, while those of all the other unknowns vanish.

$$\therefore DX^s = k_1 A_1^s + k_2 A_2^s + \dots + k_m A_m^s$$

But the right hand member of this equation is what D becomes when the coefficients $a_1^s, a_2^s, \dots, a_m^s$ of the unknown x^s are replaced by the absolute terms k_1, k_2, \dots, k_m in order.

Hence,

$$X = \frac{\begin{vmatrix} a_1' & \dots & a_1^{(s-1)} & k_1 & a_1^{(s+1)} & \dots & a_1'' \\ a_2' & \dots & a_2^{(s-1)} & k_2 & a_2^{(s+1)} & \dots & a_2'' \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_m' & \dots & a_m^{(s-1)} & k_m & a_m^{(s+1)} & \dots & a_m'' \end{vmatrix}}{\begin{vmatrix} a_1' & a_1' & \dots & a_1'' \\ a_2' & a_2' & \dots & a_2'' \\ \dots & \dots & \dots & \dots \\ a_m' & a_m' & \dots & a_m'' \end{vmatrix}}$$

This result may be summarized as follows:

(a) The common denominator of the fractions expressing the value of the unknowns in a system of n linear equations involving n unknowns is the determinant of the coefficients, these being written in the same order as in the given equations.

(b) The numerator of the fraction giving the value of any one of the unknowns is a determinant which may be formed from the determinant of the coefficients by substituting for the column made up of the coefficients of the unknown in question

a column whose elements are the absolute terms of the equations taken in the same order as the coefficients which they replace.

Illustration;

$$\begin{cases} x + y - 2z = 7 \\ 2x - 3y - 2z = 0 \\ x - 2y - 3z = 3 \end{cases}$$

$$X = \frac{\begin{vmatrix} 7 & 1 & -2 \\ 0 & -3 & -2 \\ 3 & -2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -2 \\ 2 & -3 & -2 \\ 1 & -2 & -3 \end{vmatrix}} = \frac{63 - 6 - 18 - 28}{9 - 2 + 8 - 6 + 6 - 4} = \frac{11}{11} = 1$$

$$Y = \frac{\begin{vmatrix} 1 & 7 & -2 \\ 2 & 0 & -2 \\ 1 & 3 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -2 \\ 2 & -3 & -2 \\ 1 & -2 & -3 \end{vmatrix}} = \frac{-14 - 12 + 42 + 6}{11} = \frac{22}{11} = 2$$

$$Z = \frac{\begin{vmatrix} 1 & 1 & 7 \\ 2 & -3 & 0 \\ 1 & -2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -2 \\ 2 & -3 & -2 \\ 1 & -2 & -3 \end{vmatrix}} = \frac{-9 - 28 + 21 - 6}{11} = \frac{-22}{11} = -2$$

Consistency of Linear Systems

If the number of given equations is greater than the number of unknowns, their consistency with one another depends upon some relation among the coefficients. If there are $(n+1)$ equations involving n unknowns, let the equations be

[illegible]

In order that the above equations be consistent, the values of the unknowns obtained by solving any n of them must satisfy the remaining equation. Solving the first n equations and substituting the values of x' , x'' , x in the last equation, and clearing of fractions, the result reduces to

$$E = \begin{vmatrix} a'_1 & \dots & a''_1 k_1 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a'_n & \dots & a''_n k_n \\ a'_{n+1} & \dots & a''_{n+1} k_{n+1} \end{vmatrix} = 0$$

This must be true if the given equations are to be consistent. Hence the condition of consistency for a set of linear equations involving a number of unknowns one less than the number of equations is that the determinant of the coefficients and

is expressed by the following rectangular array, or matrix:

$$\begin{vmatrix} a_1' & \dots & a_n' & \dots & a_n' \\ \dots & \dots & \dots & \dots & \dots \\ a_1'' & \dots & a_n'' & \dots & a_n'' \\ k_1 & \dots & k_n & \dots & k_n \end{vmatrix} = 0$$

Homogeneous Linear Systems

Let the system of linear homogeneous equations be written

$$\begin{cases} a_1' x' + a_1'' x'' + \dots + a_1'' x'' = 0 \\ a_2' x' + a_2'' x'' + \dots + a_2'' x'' = 0 \\ \dots \\ a_n' x' + a_n'' x'' + \dots + a_n'' x'' = 0 \end{cases}$$

Now the general solution becomes

$$x^s = 0/E$$

where E is the determinant of the coefficients. That is, all the unknowns are equal to zero, unless E also equals zero. Then the value of each unknown is in the determinant form 0/0. However, the ratios of the unknowns may be obtained. If we divide each of the given equations by any one of the unknowns, as x^s , the system then becomes

From the last two equations

$$\frac{f(x)}{p_1 x + p_0} = \frac{\phi(x)}{q_1 x + q_0}$$

$$\therefore (q_1 x + q_0)(a_2 x^2 + a_1 x + a_0) \equiv (p_1 x + p_0)(b_2 x^2 + b_1 x + b_0).$$

Equating the coefficients of like powers of x , we have the equations

$$\begin{cases} q_1 a_2 + 0 - p_1 b_2 + 0 = 0 \\ q_1 a_1 + q_0 a_2 - p_1 b_1 - p_0 b_2 = 0 \\ q_1 a_0 + q_0 a_1 - p_1 b_0 - p_0 b_1 = 0 \\ 0 + q_0 a_0 + 0 - p_0 b_0 = 0 \end{cases}$$

Hence, the eliminant is

$$E = \begin{vmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{vmatrix} = 0$$

In general, let

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + \dots + a_1 x + a_0 = 0$$

$$\phi(x) = b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0 = 0$$

If r is the common root of the two equations, let

$$\frac{f(x)}{x-r} = p_{m-1} x^{m-1} + p_{m-2} x^{m-2} + \dots + p_1 x + p_0 \equiv f_1(x)$$

$$\frac{\phi(x)}{x-r} = q_{n-1} x^{n-1} + q_{n-2} x^{n-2} + \dots + q_1 x + q_0 \equiv \phi_1(x)$$

in which the coefficients $p_{m-1}, \dots, p_0, q_{m-1}, \dots, q_0$ are undetermined. Then

$$f_1(x) \phi(x) \equiv \phi_1(x) f(x)$$

If we expand this identity and equate the coefficients of like powers of x we will have $(m+n)$ homogeneous equations involving the $(m+n)$ coefficients. Hence the determinant of the system must vanish if the given equations have a common root, and the eliminant is this determinant.

Sylvester's Dialytic Method of Elimination

To eliminate the unknowns from the two equations

$$\left. \begin{aligned} a_3 x^3 + a_2 x^2 + a_1 x + a_0 &= 0 \\ b_2 x^2 + b_1 x + b_0 &= 0 \end{aligned} \right\}.$$

Multiply the first equation by x , and the second by x and x^2 successively. The result is a system of five equations in the four unknowns x, x^2, x^3, x^4 .

$$\left. \begin{aligned} a_3 x^3 + a_2 x^2 + a_1 x + a_0 &= 0 \\ a_3 x^4 + a_2 x^3 + a_1 x^2 + a_0 x &= 0 \\ b_2 x^2 + b_1 x + b_0 &= 0 \\ b_2 x^3 + b_1 x^2 + b_0 x &= 0 \\ b_2 x^4 + b_1 x^3 + b_0 x^2 &= 0 \end{aligned} \right\}$$

If these equations are consistent the eliminant E must vanish.

$$E = \begin{vmatrix} 0 & a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 \\ 0 & b_2 & b_1 & b_0 & 0 \\ b_2 & b_1 & b_0 & 0 & 0 \end{vmatrix} = 0$$

Since the equations are consistent, a solution of four of the five equations in four unknowns will be a solution of the given equations.

This method may be generalized. Let the two given equations be

$$\left. \begin{aligned} a_m x^m + \dots + a_1 x + a_0 &= 0 \\ b_n x^n + \dots + b_1 x + b_0 &= 0 \end{aligned} \right\}$$

If the first equation is multiplied $(n-1)$ times in succession by x and the second $(m-1)$ times, $(m+n)$ equations are obtained involving as unknowns the first $(m+n-1)$ powers of x . The eliminant of these equations is a determinant of the order of $(m+n)$.

The same method may be used to eliminate one or both variables from a pair of homogeneous equations.

Illustration:

$$\left. \begin{aligned} 2x^3 - 5x^2y - 9y^3 &= 0 \\ 3x^2 - 7xy - 6y^2 &= 0 \end{aligned} \right\}$$

If we divide the first equation by y^3 and the second by y^2 and then multiply the first by $\frac{x}{y}$ and the second twice in succession by $\frac{x}{y}$, we get five equations in the four unknowns

$\frac{x}{y}$, $\frac{x^2}{y^2}$, $\frac{x^3}{y^3}$, $\frac{x^4}{y^4}$, as follows:

$$\left. \begin{aligned} 2 \frac{x^3}{y^3} - 5 \frac{x^2}{y^2} - 9 &= 0 \\ 2 \frac{x^4}{y^4} - 5 \frac{x^3}{y^3} - 9 \frac{x}{y} &= 0 \\ 3 \frac{x^2}{y^2} - 7 \frac{x}{y} - 6 &= 0 \\ 3 \frac{x^3}{y^3} - 7 \frac{x^2}{y^2} - 6 \frac{x}{y} &= 0 \\ 3 \frac{x^4}{y^4} - 7 \frac{x^3}{y^3} - 6 \frac{x^2}{y^2} &= 0 \end{aligned} \right\}$$

Eliminating, we get

$$E = \begin{vmatrix} 0 & 2 & -5 & 0 & -9 \\ 2 & -5 & 0 & -9 & 0 \\ 0 & 0 & 3 & -7 & -6 \\ 0 & 3 & -7 & -6 & 0 \\ 3 & -7 & -6 & 0 & 0 \end{vmatrix} = 0$$

Hence, the two equations are consistent.

Discriminant of an Equation

The discriminant of an equation involving a single variable is the simplest function of the coefficients, in a rational and integral form, whose vanishing expresses the condition for equal roots.

Given:

$$f(x) \equiv a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

The first derivative of $f(x)$ is

$$f'(x) \equiv n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + a_1 = 0$$

The eliminant E of $f(x) = 0$ and $f'(x) = 0$ is called the discriminant of the equation $f(x) = 0$, since if E vanishes, $f(x) = 0$ and $f'(x) = 0$ have a common root and hence $f(x) = 0$ has equal roots.

Forming the eliminant of these two equations by Sylvester's method, we have

$$E = \begin{vmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_0 & 0 & 0 & 0 & \dots \\ 0 & a_n & a_{n-1} & \dots & a_1 & a_0 & 0 & 0 & \dots \\ 0 & 0 & a_n & \dots & a_2 & a_1 & a_0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n a_n & (n-1) a_{n-1} & (n-2) a_{n-2} & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & n a_n & (n-1) a_{n-1} & \dots & a_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & n a_n & \dots & 2 a_2 & a_1 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

in which the first $(n-1)$ rows are formed from the coefficients of $f(x) = 0$ and the last n rows from the coefficients of $f'(x) = 0$

If we multiply the first row of E by n and subtract it from the n th row, the first element of the n th row becomes zero. Hence E is easily reducible to a determinant of the order $(2n - 2)$ multiplied by a_n .

The discriminant of an equation may also be found as follows:

$$f(x) = 0 \quad \text{and} \quad f'(x) = 0$$

being simultaneous equations when $f(x) = 0$ has equal roots, the equation

$$n f(x) - x f'(x) = 0$$

is also consistent with them and is of the $(n-1)$ th degree. Since $f'(x) = 0$ is also of the $(n-1)$ th degree, the eliminant of

$$\left. \begin{array}{l} n f(x) - x f'(x) = 0 \\ \text{and } f'(x) = 0 \end{array} \right\}$$

is also the discriminant Δ , a determinant of the order $(2n-2)$.

For an illustration we shall find the discriminant of the cubic

$$a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0.$$

The eliminant of the equations

$$\left. \begin{array}{l} a_1 x^2 + 2 a_2 x + 3 a_0 = 0 \\ 3 a_3 x^2 + 2 a_2 x + a_1 = 0 \end{array} \right\}$$

is the discriminant Δ of the given cubic.

$$\Delta = \begin{vmatrix} a_1 & 2 a_2 & 3 a_0 & 0 \\ 0 & a_1 & 2 a_2 & 3 a_0 \\ 3 a_3 & 2 a_2 & a_1 & 0 \\ 0 & 3 a_3 & 2 a_2 & a_1 \end{vmatrix} = 0$$

As a special case, take the typical quadratic

$$a x^2 + b x + c = 0.$$

Solving $n f(x) - x f'(x) = 0$ with $f'(x) = 0$

$$\left. \begin{aligned} b x + 2 c &= 0 \\ 2 a x + b &= 0 \end{aligned} \right\}$$

$$\Delta = \begin{vmatrix} b & 2c \\ 2a & b \end{vmatrix} = b^2 - 4ac,$$

which is the well known expression for determining the character of the roots. If $\Delta = 0$, the roots are both real and equal; If Δ is positive the roots are both real, and rational or irrational according as Δ is or is not a perfect square. If Δ is negative, the roots are both imaginary.

Some special Solutions of Simultaneous Quadratics

Certain types of simultaneous quadratics may be solved readily by determinants. We shall consider here three of these types.

Type I Find x and y in

$$\left. \begin{aligned} \frac{a_1 x + b_1 y}{a_2 x + b_2 y} &= \frac{m_1}{m_2} \\ x^2 + y^2 &= r^2 \end{aligned} \right\}$$

Let f be a factor such that

$$\left. \begin{aligned} a_1 x + b_1 y &= f m_1 \\ a_2 x + b_2 y &= f m_2 \end{aligned} \right\}$$

Solving these two equations

$$x = \frac{f \begin{vmatrix} m_1 & b_1 \\ m_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \equiv \frac{f D}{\Delta}; \quad y = \frac{f \begin{vmatrix} a_1 & m_1 \\ a_2 & m_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \equiv \frac{f D_1}{\Delta}.$$

Substituting in the second of the given equations,

$$f^2 D^2 + f^2 D_1^2 = r^2 \Delta^2 \quad \therefore f = \frac{r \Delta}{\pm \sqrt{D^2 + D_1^2}};$$

$$\therefore X = \frac{r D}{\pm \sqrt{D^2 + D_1^2}}; \quad Y = \frac{r D_1}{\pm \sqrt{D^2 + D_1^2}}.$$

Type II

Solve the equations

$$\left. \begin{aligned} a_1 x + b_1 y &= m_1 x y \\ a_2 x + b_2 y &= m_2 x y \end{aligned} \right\}.$$

Divide these equations and as before let

$$\left. \begin{aligned} a_1 x + b_1 y &= f m_1 \\ a_2 x + b_2 y &= f m_2 \end{aligned} \right\}$$

$$\therefore X = \frac{f |m, b_2|}{|a, b_2|}; \quad Y = \frac{f |a, m_2|}{|a, b_2|}.$$

Substituting these values of X and Y in the first of the given equations and solving for f,

$$f = \frac{|a, b_2| [a, |m, b_2| + b, |a, m_2|]}{m, |m, b_2| |a, m_2|},$$

$$\therefore X = \frac{|a, b_2|}{|a, m_2|}; \quad Y = \frac{|a, b_2|}{|m, b_2|}.$$

Type III

Solve the equations

$$\left. \begin{aligned} a x^2 + b x y + c y^2 &= d \\ e x^2 + f x y + g y^2 &= h \end{aligned} \right\} \quad (1)$$

These equations may be written

$$\left. \begin{aligned} x^2 + 2 a_1 x y + b_1 y^2 &= m_1 \\ x^2 + 2 a_2 x y + b_2 y^2 &= m_2 \end{aligned} \right\} \quad (2)$$

Write the equations in (2) in the form

$$\left. \begin{aligned} x(x + a_1 y) + y(a_1 x + b_1 y) &= m_1 \\ x(x + a_2 y) + y(a_2 x + b_2 y) &= m_2 \end{aligned} \right\} \quad (3)$$

Then

$$X = \frac{\begin{vmatrix} m_1 & a_1 x + b_1 y \\ m_2 & a_2 x + b_2 y \end{vmatrix}}{\Delta}; \quad Y = \frac{\begin{vmatrix} x + a_1 y & m_1 \\ x + a_2 y & m_2 \end{vmatrix}}{\Delta}$$

$$\text{Where } \Delta = \begin{vmatrix} x + a_1 y & a_1 x + b_1 y \\ x + a_2 y & a_2 x + b_2 y \end{vmatrix}$$

From these values of X and Y we have the equations

$$(\Delta + |a_1, m_2|) \cdot x + |b_1, m_2| \cdot y = 0$$

$$(m_2 - m_1) \cdot x + (|a_1, m_2| - \Delta) \cdot y = 0$$

For these equations to be consistent, their determinants must vanish, i. e.,

$$\begin{vmatrix} \Delta + |a_1, m_2| & |b_1, m_2| \\ m_2 - m_1 & |a_1, m_2| - \Delta \end{vmatrix} = 0$$

Solving the quadratic for Δ ,

$$\Delta = \pm \sqrt{|a, m_2|^2 - |b, m_2|(m_1 - m_2)}.$$

The above value of X may be written in the form

$$X = \frac{|m, a_2| \cdot x + |m, b_2| \cdot y}{\Delta}$$

which reduces to

$$X = \frac{|m, b_2| \cdot y}{\Delta + |a, m_2|}$$

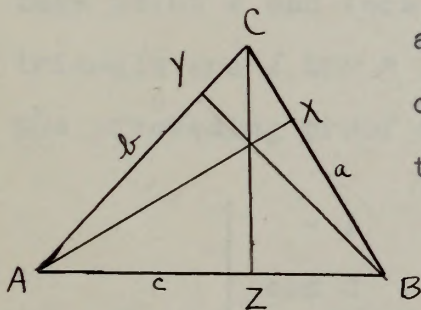
Substituting this value in the first of equations (2), we have

$$\frac{|m, b_2|^2 y^2}{(\Delta + |a, m_2|)^2} + \frac{2a, |m, b_2| y^2}{\Delta + |a, m_2|} + b, y^2 = m,$$

a pure quadratic which may be solved for Y.

2. Applications to Trigonometry

Relation between cosines of three angles of a triangle



Let A, B, C, be the three vertices of a triangle, and a, b, c, the sides respectively opposite. Draw the three altitudes, meeting the sides at x, y, z. Obviously,

$$a = CX + XB$$

$$\text{or } a = b \cos C + c \cos B.$$

Now we can form the three equations in a, b, c :

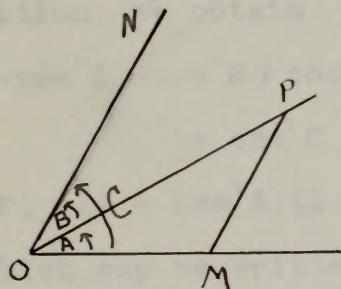
$$\left. \begin{aligned} -a + b \cos C + c \cos B &= 0 \\ a \cos C - b + c \cos A &= 0 \\ a \cos B + b \cos A - c &= 0 \end{aligned} \right\}$$

These equations are consistent, and hence their determinant vanishes, and we have the following relation:

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0$$

$$\text{or, } \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cdot \cos B \cdot \cos C - 1 = 0.$$

Condition that three lines from a point be in one plane



Let lines be OM, ON, OP

$$\text{Let } \angle MOP = A$$

$$\angle PON = B$$

$$\angle MON = C$$

From some point M in OM, draw a parallel to ON. If OP is in the same plane with OM and ON, this parallel will meet OP at some point P and form the triangle MOP. The angles of this triangle are $\angle MOP = A$, $\angle OPM = B$, $\angle OMP = \pi - C$. Hence from the preceding proof we have the relation

$$\begin{vmatrix} -1 & -\cos C & \cos B \\ -\cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0.$$

If now we multiply the first and second rows by -1 and then the resulting third column by -1 , we obtain the following as the required condition:

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

or this may be written

$$1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C = 0$$

Conversely, if this condition is fulfilled, then the three lines are in a plane.

If we both add and subtract $\cos^2 A \cos^2 B$ to the given condition, we obtain

$$1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B - \cos^2 A \cos^2 B + 2 \cos A \cos B \cos C - \cos^2 C = 0$$

$$\text{or, } (1 - \cos^2 A)(1 - \cos^2 B) - (\cos A \cos B - \cos C)^2 = 0,$$

which may be written

$$\sin^2 A \sin^2 B - (\cos A \cos B - \cos C)^2 = 0,$$

or,

$$(\sin A \sin B + \cos A \cos B - \cos C)(\sin A \sin B - \cos A \cos B + \cos C) = 0$$

The first factor is equal to

$$\cos(A-B) - \cos C = 2 \sin \frac{C+A-B}{2} \sin \frac{C-A+B}{2},$$

and the second factor is equal to

$$\cos C - \cos(A+B) = 2 \sin \frac{A+B+C}{2} \sin \frac{A+B-C}{2}.$$

Multiplying these gives the equation

$$4 \sin \frac{A+B+C}{2} \sin \frac{C+B-A}{2} \sin \frac{C+A-B}{2} \sin \frac{A+B-C}{2} = 0$$

For this equation to hold, one of the three angles A, B, or C, must be equal to the other two, ~~Such~~ as $C = A+B$. Hence the three lines, OM, ON, OP, are in the same plane.

If we consider the lines OM and ON as the coordinate axes, the angle C is 90° and the above relation reduces to

$$\begin{vmatrix} 1 & \cos A & \cos B \\ \cos A & 1 & 0 \\ \cos B & 0 & 1 \end{vmatrix} = 0,$$

or, $\cos^2 A + \cos^2 B = 1$

Relation between three sides of a triangle
and the angle opposite one of them

As above, we have the equations

$$\left. \begin{array}{rrcr} -a & +b \cos C & +c \cos B & = 0 \\ a \cos C & -b & +c \cos A & = 0 \\ a \cos B & +b \cos A & -c & = 0 \end{array} \right\}$$

Consider the angles B and C as the unknowns. The above equations may be written

$$\left. \begin{array}{rrrr} a & -b \cos C & -c \cos B & = 0 \\ b - c \cos A & -a \cos C & -0 \cos B & = 0 \\ c - b \cos A & -0 \cos C & -a \cos B & = 0 \end{array} \right\}$$

Substituting these values in the equation

$$a \sin \frac{A+B+C}{2} = b \sin \frac{A+B-C}{2} = c \sin \frac{A-B+C}{2} = a \sin \frac{A+B-C}{2}$$

For this equation to hold, one of the three angles A, B, or C must be equal to the other two. Such as $C = A + B$, hence the three lines, AB, AC, and BC, are in the same plane.

It is assumed the lines AB and AC are coordinate axes. The angle C is 90° and the above relation reduces to

$$\cos^2 A + \cos^2 B = 1$$

$$\cos^2 A + \cos^2 B = 1$$

Relation between three sides of a triangle and the angle opposite one of them

As above, we have the equations

$$\begin{cases} a \cos B = b \cos A \\ b \cos C = c \cos B \\ c \cos A = a \cos C \end{cases}$$

Consider the angles B and C as the unknowns. The above equations may be written

$$\begin{cases} a - b \cos A = c \cos B \\ b - c \cos B = a \cos C \\ c - a \cos C = b \cos A \end{cases}$$

Eliminating $-\cos B$ and $-\cos C$, we have the following

$$\begin{vmatrix} a & b & c \\ (b - c \cos A) & a & 0 \\ (c - b \cos A) & 0 & a \end{vmatrix} = 0.$$

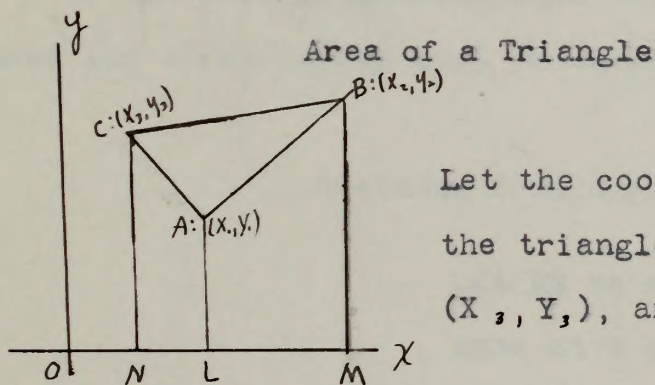
If the last two columns of this determinant are divided by a and the resulting first line multiplied by a , we have

$$\begin{vmatrix} a^2 & b & c \\ (b - c \cos A) & 1 & 0 \\ (c - b \cos A) & 0 & 1 \end{vmatrix} = 0$$

Expanding this we have the well known relation

$$a^2 = b^2 + c^2 - 2bc \cos A$$

4. Applications to Analytic Geometry



Let the coordinates of the vertices of the triangle ABC be (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , and denote the area by Δ .

It is evident from the figure that

$$\begin{aligned} \Delta &= \text{trapezoid BN} - \text{trapezoid BL} - \text{trapezoid CL} \\ &= \frac{1}{2} (y_2 + y_3) (x_2 - x_3) - \frac{1}{2} (y_2 + y_1) (x_2 - x_1) - \frac{1}{2} (y_3 + y_1) (x_1 - x_3) \end{aligned}$$

Expanding gives

$$2 \Delta = y_3 x_2 - y_2 x_3 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

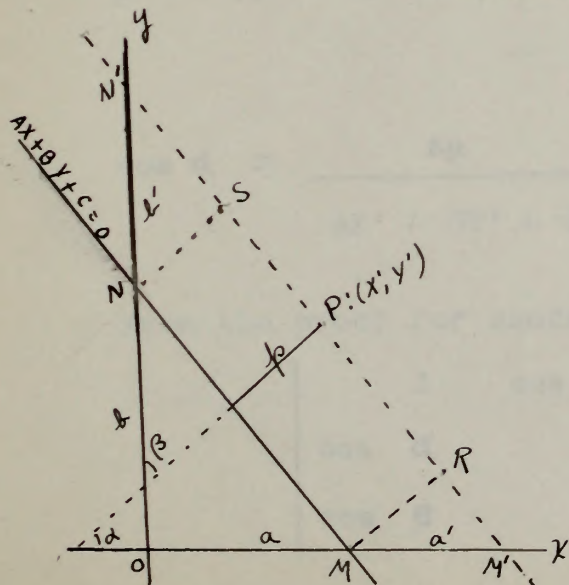
If the first row is multiplied by x , and subtracted from the second, then by y , and subtracted from the third, the determinant reduces to

$$2 \Delta = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}.$$

It must be noticed that the area of the triangle changes sign if the cyclical order of the letters is changed. The counter-clockwise order gives a positive area.

If the determinant vanishes the area is equal to zero, and hence the three points must be collinear.

Distance from a point to a line



Let MN be a line cutting the X and Y axes at M and N respectively, and (x', y') the coordinates of some point P . Let p be the distance from P to MN , and α and β the angles made by p with the axes. Draw $M'N'$ through P parallel to MN , also MR and NS parallel to p . Let a, b, a', b' , be the intercepts of MN and $M'N'$.

Let the equation of MN be

$$AX + BY + C = 0.$$

Then since $M'N'$ is parallel to MN , the equation of $M'N'$ is

$$AX + BY + C' = 0.$$

But $C' = -(AX' + BY')$, and hence ^{the} equation of $M'N'$ may be written

$$AX + BY = AX' + BY'.$$

The intercepts, a , b , a' , b' , are:

$$a = -\frac{C}{A}$$

$$a' = \frac{AX' + BY'}{A}$$

$$b = -\frac{C}{B}$$

$$b' = \frac{AX' + BY'}{B}$$

$$MR = MM' \cos RMM'$$

$$NS = NN' \cos SNN'$$

$$p = (a' - a) \cos \alpha = \frac{AX' + BY' + C}{A} \cos \alpha$$

$$p = (b' - b) \cos \beta = \frac{AX' + BY' + C}{B} \cos \beta$$

$$\cos \alpha = \frac{Ap}{AX' + BY' + C}, \quad \cos \beta = \frac{Bp}{AX' + BY' + C}.$$

From the proof for condition that three lines lie in one plane,

$$\begin{vmatrix} 1 & \cos \alpha & \cos \beta \\ \cos \alpha & 1 & 0 \\ \cos \beta & 0 & 1 \end{vmatrix} = 0.$$

Substituting in the above equation for $\cos \alpha$ and $\cos \beta$, we have

$$\begin{vmatrix}
 1 & \frac{Ap}{AX' + BY' + C} & \frac{Bp}{AX' + BY' + C} \\
 \frac{Ap}{AX' + BY' + C} & 1 & 0 \\
 \frac{Bp}{AX' + BY' + C} & 0 & 1
 \end{vmatrix} = 0$$

Multiplying the first row and then the first column by

$\frac{AX' + BY' + C}{p}$ gives

$$\begin{vmatrix}
 \frac{(AX' + BY' + C)^2}{p^2} & A & B \\
 A & 1 & 0 \\
 B & 0 & 1
 \end{vmatrix} = 0.$$

Expanding,

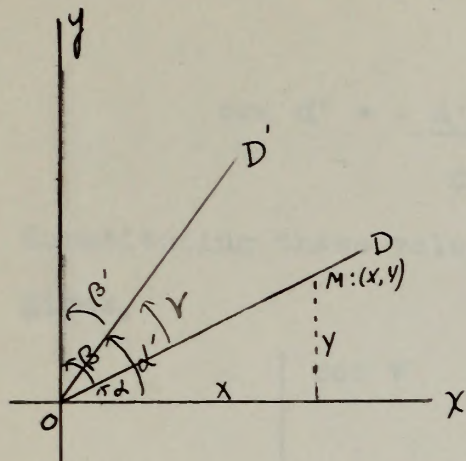
$$\frac{(AX' + BY' + C)^2}{p^2} - B^2 - A^2 = 0.$$

$$\therefore p = \frac{AX' + BY' + C}{\pm \sqrt{A^2 + B^2}}.$$

As a special case, we have the formula for the distance from the origin to a line by substituting $x' = 0$ and $y' = 0$ in the above, giving

$$p = \frac{C}{\pm \sqrt{A^2 + B^2}}.$$

Angle between two lines in terms of inclinations



Let OD and OD' be two lines from the origin. α and β are the angles made by OD with the axes, and α' and β' the angles made by OD' with the axes. γ is the angle between OD and OD'. On OD take OM = 1 and let coordinates of M be (x, y).

We can now write the following equations:

$$\left. \begin{aligned} \cos \gamma &= x \cos \alpha' + y \cos \beta' \\ \cos \alpha &= x \cdot 1 + y \cdot 0 \\ \cos \beta &= x \cdot 0 + y \cdot 1 \end{aligned} \right\} = 0$$

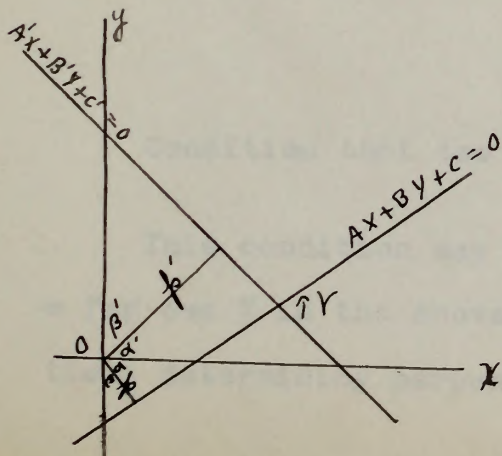
Eliminating -x and -y gives the required relation,

$$\begin{vmatrix} \cos \gamma & \cos \alpha' & \cos \beta' \\ \cos \alpha & 1 & 0 \\ \cos \beta & 0 & 1 \end{vmatrix} = 0$$

which reduces to

$$\cos \gamma = \cos \alpha \cos \alpha' + \cos \beta \cos \beta'$$

Angle between two lines in terms of their equations



Let the two lines be

$$\begin{cases} AX + BY + C = 0 \\ A'X + B'Y + C' = 0 \end{cases}$$

Let p and p' be the perpendicular distances from the origin to these lines; α and β , α' and β' the angles made by them with the coordinate axes, and γ the angle between the lines.

$$\cos \alpha = -\frac{A p}{C}$$

$$\cos \beta = -\frac{B p}{C}$$

$$\cos \alpha' = -\frac{A' p'}{C'}$$

$$\cos \beta' = -\frac{B' p'}{C'}$$

Substituting these values in the determinant of the last proof gives,

$$\begin{vmatrix} \cos \gamma & -\frac{A' p'}{C'} & -\frac{B' p'}{C'} \\ -\frac{A p}{C} & 1 & 0 \\ -\frac{B p}{C} & 0 & 1 \end{vmatrix} = 0$$

from which we get the following equation:

$$\begin{aligned} \cos \gamma &= \left(\frac{A p}{C} \times \frac{A' p'}{C'} \right) + \left(\frac{B p}{C} \times \frac{B' p'}{C'} \right) \\ &= \frac{p p'}{C C'} (AA' + BB'). \end{aligned}$$

(page 35)

If we substitute the values of p and p' already derived, this becomes

$$\cos \gamma = \frac{AA' + BB'}{\sqrt{(A^2 + B^2)(A'^2 + B'^2)}}.$$

Condition that two lines be perpendicular

This condition may be obtained immediately by substituting 0 for $\cos \gamma$ in the above relations, giving the following equations determining perpendicularity:

$$(1) \begin{vmatrix} 0 & \cos \alpha' & \cos \beta' \\ \cos \alpha & 1 & 0 \\ \cos \beta & 0 & 1 \end{vmatrix} = 0$$

which is expanded

$$\cos \alpha \cos \alpha' + \cos \beta \cos \beta' = 0.$$

$$(2) \begin{vmatrix} 0 & A' & B' \\ A & 1 & 0 \\ B & 0 & 1 \end{vmatrix} = 0$$

or $A A' + B B' = 0$

which may also be written

$$\frac{A}{B'} + \frac{B}{A'} = 0.$$

Equation of line through two given points

Let line be

$$C + A X + B Y = 0$$

This line will pass through two points (X', Y') and (X'', Y'') if the following equations hold

$$C + A X' + B Y' = 0$$

$$C + A X'' + B Y'' = 0$$

Consider these as three linear homogeneous equations in the coefficients C, A, B . In order that they be consistent, the determinant must vanish, i. e.,

$$\begin{vmatrix} 1 & X & Y \\ 1 & X' & Y' \\ 1 & X'' & Y'' \end{vmatrix} = 0$$

which is the equation of the required line.

This equation also expresses the condition that three points (X, Y) , (X', Y') , (X'', Y'') be collinear.

Condition that three lines are concurrent

Let the equations of the three lines be

$$AX + BY + C = 0$$

$$A'X + B'Y + C' = 0$$

$$A''X + B''Y + C'' = 0$$

If they all pass through the same point (X', Y') , the three equations must be satisfied when the coordinates of this point are substituted in the given equations; i. e., the following equations must hold:

$$AX' + BY' + C = 0$$

$$A'X' + B'Y' + C' = 0$$

$$A''X' + B''Y' + C'' = 0$$

If these equations are consistent, the determinant must vanish. Hence, the required condition is

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0$$

This equation also expresses the condition that three points (X, Y) , (X', Y') , (X'', Y'') be collinear.

Condition that three lines are concurrent

Let the equations of the three lines be

$$AX + BY + C = 0$$

$$A'X + B'Y + C' = 0$$

$$A''X + B''Y + C'' = 0$$

If they all pass through the same point (X, Y) , the three equations must be satisfied when the coordinates of this point are substituted in the given equations; i. e., the following equations must hold:

$$AX + BY + C = 0$$

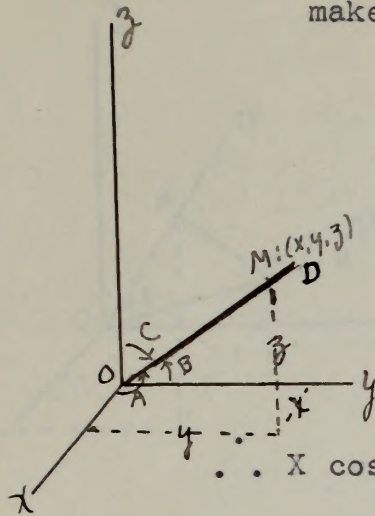
$$A'X + B'Y + C' = 0$$

$$A''X + B''Y + C'' = 0$$

If these equations are consistent, the determinant must vanish. Hence, the required condition is

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0$$

Relation between three angles that a line in space makes with coordinate axes



Let OD be any line from the origin. Take $OM = 1$ and let coordinates of M be (X, Y, Z) . Let angles made with axes be A, B, C.

$$\cos A = \frac{X}{OM} = X, \quad \cos B = Y, \quad \cos C = Z.$$

$$\therefore X \cos A + Y \cos B + Z \cos C = X^2 + Y^2 + Z^2 = \overline{OM} = 1.$$

Hence the following equations are true:

$$\begin{array}{rclcl} -1 + X \cos A + Y \cos B + Z \cos C & = & 0 \\ -\cos A + X & + Y \cdot 0 & + Z \cdot 0 & = & 0 \\ -\cos B + X \cdot 0 & + Y & + Z \cdot 0 & = & 0 \\ -\cos C + X \cdot 0 & + Y \cdot 0 & + Z & = & 0 \end{array}$$

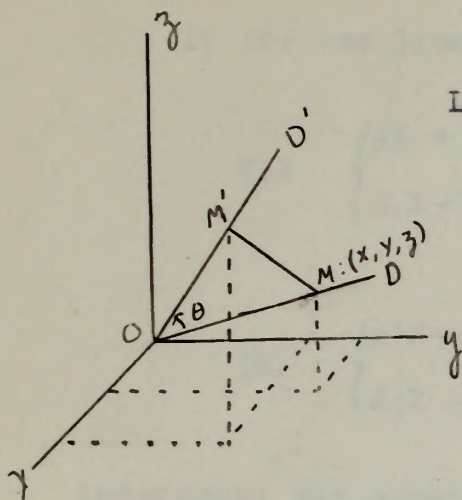
Eliminating the three variables X, Y, Z, from the above equations, gives

$$\begin{vmatrix} 1 & \cos A & \cos B & \cos C \\ \cos A & 1 & 0 & 0 \\ \cos B & 0 & 1 & 0 \\ \cos C & 0 & 0 & 1 \end{vmatrix} = 0$$

which when expanded becomes

$$\cos^2 A + \cos^2 B + \cos^2 C = 1.$$

Angle between two lines in space



Let OD and OD' be two lines from origin,
A, B, C, are three angles made by OD
with axes.

A', B', C' are three angles made by OD'
with axes.

θ is the angle between OD and OD'

On OD take OM = 1. Draw MM' \perp OD'

As in the similar proof on **Page 40**, we have the following equations:

$$\begin{aligned}
 -\cos \theta + X \cos A' + Y \cos B' + Z \cos C' &= 0 \\
 -\cos A + X + Y \cdot 0 + Z \cdot 0 &= 0 \\
 -\cos B + X \cdot 0 + Y + Z \cdot 0 &= 0 \\
 -\cos C + X \cdot 0 + Y \cdot 0 + Z &= 0
 \end{aligned}$$

Eliminating X, Y, Z, gives

$$\begin{vmatrix}
 \cos \theta & \cos A' & \cos B' & \cos C' \\
 \cos A & 1 & 0 & 0 \\
 \cos B & 0 & 1 & 0 \\
 \cos C & 0 & 0 & 1
 \end{vmatrix} = 0$$

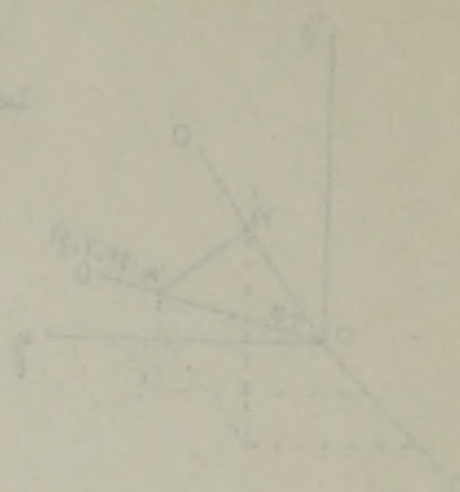
which gives the required relation.

$$\cos \theta = \cos A \cos A' + \cos B \cos B' + \cos C \cos C'$$

For the two lines to be perpendicular, the condition is obviously

$$\cos A \cos A' + \cos B \cos B' + \cos C \cos C' = 0.$$

angle between two lines in space



Let OA and OB be two lines from origin,

A, B, C are three angles made by OA

with axes.

A, B, C are three angles made by OB

with axes.

θ is the angle between OA and OB

On OB take $OC = 1$. Draw $CM \perp OA$.

As in the similar triangles OCM and OCN , we have the following

equations:

$$\begin{aligned} -\cos \theta + \cos A' + \cos B' + \cos C' &= 0 \\ -\cos A + \cos B' + \cos C' &= 0 \\ -\cos B + \cos A' + \cos C' &= 0 \\ -\cos C + \cos A' + \cos B' &= 0 \end{aligned}$$

Eliminating X, Y, Z , gives

$$\begin{vmatrix} \cos \theta & \cos A' & \cos B' & \cos C' \\ \cos A & 1 & 0 & 0 \\ \cos B & 0 & 1 & 0 \\ \cos C & 0 & 0 & 1 \end{vmatrix} = 0$$

which gives the required relation.

$$\cos \theta = \cos A \cos A' + \cos B \cos B' + \cos C \cos C'$$

For the two lines to be perpendicular, the condition is

orthogonally

$$\cos A \cos A' + \cos B \cos B' + \cos C \cos C' = 0$$

Condition that two lines intersect

If the two lines determined by the equations

$$(1) \quad \begin{cases} AX + BY + CZ + D = 0 \\ A_1X + B_1Y + C_1Z + D_1 = 0 \end{cases}$$

$$(2) \quad \begin{cases} A'X + B'Y + C'Z + D' = 0 \\ A'_1X + B'_1Y + C'_1Z + D'_1 = 0 \end{cases}$$

intersect, the equations are satisfied by the coordinates of the point of intersection. Eliminating the three variables, X, Y, Z , we obtain the required condition.

$$\begin{vmatrix} A & B & C & D \\ A_1 & B_1 & C_1 & D_1 \\ A' & B' & C' & D' \\ A'_1 & B'_1 & C'_1 & D'_1 \end{vmatrix} = 0$$

Suppose the two lines are determined by the two systems of equations

$$(1) \quad \begin{cases} X = AZ + P \\ Y = BZ + Q \end{cases}$$

$$(2) \quad \begin{cases} X = A'Z + P' \\ Y = B'Z + Q' \end{cases}$$

These equations may be written

$$\begin{aligned} -X + 0 \cdot Y + AZ + P &= 0 \\ 0 \cdot X - Y + BZ + Q &= 0 \\ -X + 0 \cdot Y + A'Z + P' &= 0 \\ 0 \cdot X - Y + B'Z + Q' &= 0 \end{aligned}$$

Eliminating X, Y, Z, we have the following

$$\begin{vmatrix} -1 & 0 & A & P \\ 0 & -1 & B & Q \\ -1 & 0 & A' & P' \\ 0 & -1 & B' & Q' \end{vmatrix} = \begin{vmatrix} 1 & 0 & A & P \\ 0 & 1 & B & Q \\ 1 & 0 & A' & P' \\ 0 & 1 & B' & Q' \end{vmatrix} = 0.$$

If we now subtract the first two rows from the last two, we obtain

$$\begin{vmatrix} 1 & 0 & A & P \\ 0 & 1 & B & Q \\ 0 & 0 & A'-A & P'-P \\ 0 & 0 & B'-B & Q'-Q \end{vmatrix} = \begin{vmatrix} A' - A & P' - P \\ B' - B & Q' - Q \end{vmatrix} = 0$$

which may be written

$$\frac{A - A'}{P - P'} = \frac{B - B'}{Q - Q'}.$$

Intersection of a line with a plane

Let line be $X = az + p$, $Y = bz + q$

and let plane be $Ax + By + Cz + D = 0$.

The coordinates of the point of intersection must satisfy all three equations, and the system may be written

$$\left. \begin{aligned} Cz + D + Ax + By &= 0 \\ -az - p + x + 0 \cdot y &= 0 \\ -bz - q + 0 \cdot x + y &= 0 \end{aligned} \right\}$$

Eliminating x and y we obtain

$$\begin{vmatrix} C z + D & A & B \\ -a z - p & 1 & 0 \\ -b z - q & 0 & 1 \end{vmatrix} = 0$$

which gives

$$z \begin{vmatrix} C & A & B \\ -a & 1 & 0 \\ -b & 0 & 1 \end{vmatrix} = \begin{vmatrix} -D & A & B \\ p & 1 & 0 \\ q & 0 & 1 \end{vmatrix}$$

Solving for z we have

$$z = - \frac{A p + B q + D}{A a + B b + C}$$

x and y may be found by substitutions;

$$x = - \frac{(B q + D) a + (B b + C) p}{A a + B b + C}$$

$$y = - \frac{(A p + D) b + (A a + C) q}{A a + B b + C}$$

Intersection of three planes

Let the equations of the planes be

$$A X + B Y + C Z + D = 0$$

$$A' X + B' Y + C' Z + D' = 0$$

$$A'' X + B'' Y + C'' Z + D'' = 0$$

(a) If the three planes intersect in a point, the coordinates of the point of intersection must satisfy the given equations. If we eliminate Y and Z we get the following equations in X:

$$\begin{vmatrix} AX + D & B & C \\ A'X + D' & B' & C' \\ A''X + D'' & B'' & C'' \end{vmatrix} = 0 \quad (1)$$

which may be written

$$X \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} + \begin{vmatrix} D & B & C \\ D' & B' & C' \\ D'' & B'' & C'' \end{vmatrix} = 0 \quad (2)$$

or $X(AB'C'') + (DB'C'') = 0.$

Hence we have the coordinates of the point of intersection:

$$X = - \frac{(DB'C'')}{(AB'C'')} \quad Y = - \frac{(AD'C'')}{(AB'C'')} \quad Z = - \frac{(AB'D'')}{(AB'C'')}.$$

(b) If the three planes intersect in three parallel lines, the values of the coordinates are infinite, and hence $(AB'C'') = 0$, and each of the other three determinants $(DB'C'')$, $(AD'C'')$, $(AB'D'')$ is unequal to zero.

(c) If the three planes intersect in two parallel lines, two of the planes, say the last two, are parallel. Then

$$A'' = m A', \quad B'' = m B', \quad C'' = m C' \quad \text{of (2)}$$

but $D'' \neq mD'$. The first determinant then becomes

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ m A' & m B' & m C' \end{vmatrix} = m \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A' & B' & C' \end{vmatrix}$$

which vanishes since two rows are identical.

The other three determinants are now

$$m \begin{vmatrix} D & B & C \\ D' & B' & C' \\ \frac{D''}{m} & B' & C' \end{vmatrix}, \quad m \begin{vmatrix} A & D & C \\ A' & D' & C' \\ A' & \frac{D''}{m} & C' \end{vmatrix}, \quad m \begin{vmatrix} A & B & D \\ A' & B' & D' \\ A' & B' & \frac{D''}{m} \end{vmatrix},$$

which are not equal to zero, since the intersections are parallel and the coordinates are infinite.

(d) If the three planes intersect in a straight line, the values of the coordinates of the point of intersection will be indeterminate, and we have,

$$(A B' C'') = 0, (D B' C'') = 0, (A D' C'') = 0, (A B' D'') = 0.$$

The converse of these four cases may also be proved.

Equation of plane through three given points.

If the plane $D + AX + BY + CZ = 0$ passes through the three points $P_1:(X_1, Y_1, Z_1)$, $P_2:(X_2, Y_2, Z_2)$, $P_3:(X_3, Y_3, Z_3)$ the following equations must also hold,

$$D + AX_1 + BY_1 + CZ_1 = 0$$

$$D + AX_2 + BY_2 + CZ_2 = 0$$

$$D + AX_3 + BY_3 + CZ_3 = 0$$

If these four equations are consistent, the determinant must vanish; i. e.,

$$\begin{vmatrix} 1 & X & Y & Z \\ 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \end{vmatrix} = 0.$$

This is the required equation.

This equation also expresses the necessary and sufficient condition that four points, P_1, P_2, P_3, P_4 , lie in the same plane. It will be noticed that this condition exactly corresponds to the condition that three points be collinear.

Condition that three lines are parallel to a plane.

The three lines

$$\left\{ \begin{array}{l} \frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} \\ \frac{x - p'}{a'} = \frac{y - q'}{b'} = \frac{z - r'}{c'} \\ \frac{x - p''}{a''} = \frac{y - q''}{b''} = \frac{z - r''}{c''} \end{array} \right.$$

will be parallel to the same plane

$$Ax + By + Cz + D = 0$$

if the following equations hold:

$$A a + B b + C c = 0$$

$$A a' + B b' + C c' = 0$$

$$A a'' + B b'' + C c'' = 0$$

For these equations to be consistent the determinant must vanish. Hence the required condition is

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0.$$

Similarly, the condition that the three planes

$$\begin{cases} A X + B Y + C Z + D = 0 \\ A' X + B' Y + C' Z + D' = 0 \\ A'' X + B'' Y + C'' Z + D'' = 0 \end{cases}$$

be parallel to the line

$$\frac{X - p}{a} = \frac{Y - q}{b} = \frac{Z - r}{c}$$

is that

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0.$$

2. Applications to the Calculus

The Jacobian

If y_1, y_2, \dots, y_m are n functions, each of the n independent variables x_1, x_2, \dots, x_m , then the determinant

$$J \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \dots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_m}{\partial x_2} \\ \dots & \dots & \dots & \dots \\ \frac{\partial y_1}{\partial x_m} & \frac{\partial y_2}{\partial x_m} & \dots & \frac{\partial y_m}{\partial x_m} \end{vmatrix}$$

is called the Jacobian of the given functions. It is often written

$$J = \frac{\partial(y_1, y_2, \dots, y_m)}{\partial(x_1, x_2, \dots, x_m)}.$$

When the functions are linear, thus:

$$y_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m$$

it follows from the above definition that the Jacobian is the determinant of the coefficients. That is

$$J \equiv (a_{11} \ a_{12} \ \dots \ a_{1m}).$$

(b) Consider the functions y_1, y_2, \dots, y_m when they are fractions having the same denominator, so that

$$y_i = \frac{u_i}{u}.$$

Differentiating and clearing of fractions

$$u^2 \frac{\partial y_i}{\partial x_k} = u \frac{\partial u_i}{\partial x_k} - u_i \frac{\partial u}{\partial x_k}.$$

Then

$$u^{2m+1} \frac{\partial (y_1 y_2 \dots y_m)}{\partial (x_1 x_2 \dots x_m)} = \begin{vmatrix} u & 0 & \dots & 0 \\ u_1 & u \frac{\partial u_1}{\partial x_1} - u_1 \frac{\partial u}{\partial x_1} & \dots & u \frac{\partial u_1}{\partial x_m} - u_1 \frac{\partial u}{\partial x_m} \\ \dots & \dots & \dots & \dots \\ u_m & u \frac{\partial u_m}{\partial x_1} - u_m \frac{\partial u}{\partial x_1} & \dots & u \frac{\partial u_m}{\partial x_m} - u_m \frac{\partial u}{\partial x_m} \end{vmatrix}$$

If we add the first column multiplied by $\frac{\partial u}{\partial x_i}$ to the $(i+1)$ th

column, we get

$$u^{2m+1} \frac{\partial (y_1 y_2 \dots y_m)}{\partial (x_1 x_2 \dots x_m)} = \begin{vmatrix} u & u \frac{\partial u}{\partial x_1} & \dots & u \frac{\partial u}{\partial x_m} \\ u_1 & u \frac{\partial u_1}{\partial x_1} & \dots & u \frac{\partial u_1}{\partial x_m} \\ \dots & \dots & \dots & \dots \\ u_m & u \frac{\partial u_m}{\partial x_1} & \dots & u \frac{\partial u_m}{\partial x_m} \end{vmatrix},$$

and dividing each of the n columns by u , gives

$$J = \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} = \frac{1}{u^{n+1}} \begin{vmatrix} u & \frac{\partial u}{\partial x_1} & \dots & \frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \dots & \frac{\partial u_1}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ u_n & \frac{\partial u_n}{\partial x_1} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}.$$

The Jacobian of Indirect Functions

(c) When u_1, u_2, \dots, u_n are each functions of y_1, y_2, \dots, y_n , these in turn being functions of x_1, x_2, \dots, x_n , then

$$J \equiv \frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = \frac{\partial(u_1, u_2, \dots, u_n)}{\partial(y_1, y_2, \dots, y_n)} \cdot \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)}.$$

This may be demonstrated by writing out the Jacobians in the right-hand member of the equation in determinant form, changing columns into rows in the first, and multiplying these two determinants. For simplicity, let $n = 2$

$$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_1} & \frac{\partial u_1}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_2} \\ \frac{\partial u_2}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix} = \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} \equiv J.$$

It follows as a particular case that

$$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \times \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = 1$$

or written in the general form

$$\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} \times \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} = 1$$

This may be regarded as a generalization of

$$\frac{dy}{dx} \neq 1 \div \frac{dx}{dy}.$$

$$y_i = a_{1i}x_1 + a_{2i}x_2 + \dots + a_{ni}x_n,$$

then since the Jacobian of linear functions is the determinant of the coefficients,

$$J = \frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = (a_{11} \ a_{12} \ \dots \ a_{1n}) \cdot \frac{\partial(u_1, u_2, \dots, u_m)}{\partial(y_1, y_2, \dots, y_m)}$$

Jacobian of Implicit Function

(e) If the functions y_1, y_2, \dots, y_n are given only as implicit functions of x_1, x_2, \dots, x_m by means of n equations

[illegible]

the Jacobian is found as follows.

From the above equations,

$$- \frac{\partial F_i}{\partial X_k} = \frac{\partial F_i}{\partial Y_1} \frac{\partial Y_1}{\partial X_k} + \frac{\partial F_i}{\partial Y_2} \frac{\partial Y_2}{\partial X_k} + \dots + \frac{\partial F_i}{\partial Y_m} \frac{\partial Y_m}{\partial X_k}$$

Using this equation and applying the rule for multiplying two determinants we have the following equation:

$$\begin{vmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \dots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \dots & \frac{\partial F_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \dots & \frac{\partial F_n}{\partial y_n} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} = (-1)^n \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_n} \end{vmatrix}$$

from which

$$J \equiv \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} = (-1)^n \frac{\partial(F_1, F_2, \dots, F_n)}{\partial(x_1, x_2, \dots, x_n)} : \frac{\partial(F_1, F_2, \dots, F_n)}{\partial(y_1, y_2, \dots, y_n)}.$$

Substituting $n = 1$ in this equation gives the well known formula

$$-\frac{\partial F_1}{\partial x_1} = \frac{\partial F_1}{\partial y_1} \frac{dy_1}{dx_1}.$$

(f) If in the system of implicit functions, F_2 does not contain x_1, \dots, x_{i-1} , then in the determinant

$$\frac{\partial(F_1, F_2, \dots, F_n)}{\partial(x_1, x_2, \dots, x_n)}$$

all elements below the principal diagonal vanish, and it reduces to

$$\frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdot \dots \cdot \frac{\partial F_n}{\partial x_n}.$$

(g) If $F_{\lambda} = -y_{\lambda} + f_{\lambda}(x_1, x_2, \dots, x_n)$, then

$$\frac{\partial (F_1, F_2, \dots, F_m)}{\partial (y_1, y_2, \dots, y_m)} = (-1)^m$$

$$\text{and } \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} = \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}.$$

(e) Page 54

(h) From the given system of implicit functions, there may be deduced by methods of Higher Analysis the following equations:

[illegible]

Since $\frac{\partial \phi_i}{\partial y_1} \frac{\partial y_1}{\partial x_h} + \dots + \frac{\partial \phi_i}{\partial y_{i-1}} \frac{\partial y_{i-1}}{\partial x_h} + \frac{\partial \phi_i}{\partial x_h} = \frac{\partial y_i}{\partial x_h}$,

we have

$$\left| \begin{array}{cccc} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} & \frac{\partial \phi}{\partial x_3} & \dots \\ 0 & \frac{\partial \phi}{\partial x_2} & \frac{\partial \phi}{\partial x_3} & \dots \\ 0 & 0 & \frac{\partial \phi}{\partial x_3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right| = \left| \begin{array}{cccc} 1 & 0 & 0 & \dots \\ -\frac{\partial \phi}{\partial y_1} & 1 & 0 & \dots \\ -\frac{\partial \phi}{\partial y_1} - \frac{\partial \phi}{\partial y_2} & 1 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right| \cdot \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)}.$$

But from the above, this equation reduces to

$$\frac{\partial(y_1, y_2, \dots, y_m)}{\partial(x_1, x_2, \dots, x_m)} = \frac{\partial \phi}{\partial x_1} \cdot \frac{\partial \phi}{\partial x_2} \dots \frac{\partial \phi}{\partial x_m}.$$

Hence if
$$\frac{\partial (y_1, y_2, \dots, y_n)}{\partial (x_1, x_2, \dots, x_n)} = 0,$$

we must have
$$\frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial \phi_2}{\partial x_2} \dots \frac{\partial \phi_n}{\partial x_n} = 0,$$

i. e., we must have some factor
$$\frac{\partial \phi_i}{\partial x_i} = 0,$$

where i is some number between 1 and n . Hence ϕ_i does not contain x_i . That is to say, we have

$$y_i = \phi_i(y_1, \dots, y_{i-1}, x_{i+1}, \dots, x_n);$$

then
$$y_{i+1} = \phi_{i+1}(y_1, \dots, y_i, x_{i+1}, \dots, x_n),$$

Eliminating x_{i+1} between these last two equations, we obtain

$$y_{i+1} = \psi_{i+1}(y_1, \dots, y_i, x_{i+2}, \dots, x_n)$$

in which y_{i+1} does not contain x_{i+1} and so on; finally, y_n is expressible as a function of the remaining $n-1$ functions,

$$y_n = \psi_n(y_1, \dots, y_{n-1}).$$

Therefore, the given functions are not independent. Hence, this proof, with the former proof that if the functions are not independent the Jacobian vanishes, gives the following theorem:

A necessary and sufficient condition that the n functions y_1, y_2, \dots, y_n , of the independent variables, x_1, x_2, \dots, x_n be functionally related is that the Jacobian vanish, i. e.,

$$J \equiv \frac{\partial (y_1, y_2, \dots, y_n)}{\partial (x_1, x_2, \dots, x_n)} = 0.$$

For an illustration, consider the equations

$$\begin{cases} u &= x + y \\ v &= x - z \\ w &= xy + xz - yz - z^2 \end{cases}$$

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & y+z \\ 1 & 0 & x-z \\ 0 & -1 & x-y-2z \end{vmatrix} = -y - z + x - z + y + 2z - x = 0;$$

since the Jacobian vanishes, the given equations are functionally related. That is, if we eliminate x , y , and z from the given equations we obtain

$$w = v \cdot (u - v).$$

The Hessian

The Jacobian of the first differential coefficients of a function of n variables is called the Hessian of the function.

Thus if y_1, y_2, \dots, y_n are the n partial derivatives

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, \text{ of a function } f(x_1, x_2, \dots, x_n),$$

the Jacobian.

$$J = \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)},$$

becomes

$$H(f) = \frac{\partial \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)}{\partial(x_1, x_2, \dots, x_n)}.$$

This may be written in determinant form

$$H(f) = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix}$$

which is the Hessian of $f(x_1, x_2, \dots, x_n)$. Since

$$\frac{\partial^2 f}{\partial x_i \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_i}$$

the Hessian is a symmetrical determinant.

If the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$, are connected by an equation with constant coefficients,

$$a_1 \frac{\partial f}{\partial x_1} + a_2 \frac{\partial f}{\partial x_2} + \dots + a_n \frac{\partial f}{\partial x_n} = 0,$$

the Hessian vanishes. The proof of this is similar to that for the Jacobian vanishing when the functions are related.

If we transform the original function $f(x_1, x_2, \dots, x_n)$ into $f'(u_1, u_2, \dots, u_n)$ by making the linear substitution,

$$x_i = a_{i1} u_1 + a_{i2} u_2 + \dots + a_{in} u_n,$$

then

$$H(f') = \frac{\partial \left(\frac{\partial f'}{\partial u_1}, \frac{\partial f'}{\partial u_2}, \dots, \frac{\partial f'}{\partial u_n} \right)}{\partial(u_1, u_2, \dots, u_n)}.$$

From the work with the Jacobian of linear functions, this reduces to

$$H(f') \equiv |a_{i,n}| \frac{\partial \left(\frac{\partial f'}{\partial u_1}, \frac{\partial f'}{\partial u_2}, \dots, \frac{\partial f'}{\partial u_n} \right)}{\partial (x_1, x_2, \dots, x_n)},$$

where $|a_{i,n}|$ is the determinant of the transformation. But

$$\frac{\partial^2 f'}{\partial x_s \partial u_k} = \frac{\partial^2 f}{\partial u_k \partial x_s},$$

therefore the last equation may be written

$$\begin{aligned} H(f') &\equiv |a_{i,n}| \frac{\partial \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)}{\partial (u_1, u_2, \dots, u_n)} \\ &\equiv |a_{i,n}|^2 \frac{\partial \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)}{\partial (x_1, x_2, \dots, x_n)} \\ \therefore H(f') &\equiv |a_{i,n}|^2 H(f). \end{aligned}$$

That is, if a function be subjected to a linear transformation, the Hessian of the transformed function will equal that of the given function multiplied by the square of the determinant of the transformation.

The Wronskian

$$a_1 f_1 + a_2 f_2 + a_3 f_3 + \dots + a_n f_n = 0, \quad (1)$$

in which a_0, a_1, \dots, a_n are not functions of x . Then if we differentiate equation (1) successively $(n-1)$ times, we have

$$\left. \begin{aligned} &a_1 f'_1 + a_2 f'_2 + a_3 f'_3 + \dots + a_m f'_m = 0 \\ &a_1 f''_1 + a_2 f''_2 + a_3 f''_3 + \dots + a_m f''_m = 0 \\ &\dots\dots\dots \\ &a_1 f^{(n-1)}_1 + a_2 f^{(n-1)}_2 + a_3 f^{(n-1)}_3 + \dots + a_m f^{(n-1)}_m = 0 \end{aligned} \right\}. \quad (2)$$

Eliminating a_1, a_2, \dots, a_m from (1) and (2), gives

$$\begin{vmatrix} f_1 & f_2 & f_3 & \dots & f_n \\ f_1' & f_2' & f_3' & \dots & f_n' \\ f_1'' & f_2'' & f_3'' & \dots & f_n'' \\ \dots & \dots & \dots & \dots & \dots \\ f_1^{n-1} & f_2^{n-1} & f_3^{n-1} & \dots & f_n^{n-1} \end{vmatrix} = D(f_1, f_2, f_3, \dots, f_n) = 0 \quad (3)$$

This determinant, $D(f_1, f_2, f_3, \dots, f_n)$ is known as the Wronskian of f_1, f_2, \dots, f_n .

If we denote the given functions by y_1, y_2, \dots, y_n , and the derivatives by y_{11}, y_{21}, \dots (i. e., the second subscript denoting the derivatives), we may write equation (3) as follows:

$$\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_{11} & y_{21} & \dots & y_{n1} \\ \dots & \dots & \dots & \dots \\ y_{1n-1} & y_{2n-1} & \dots & y_{nn-1} \end{vmatrix} \equiv D(y_1, y_2, \dots, y_n) = 0. \quad (4)$$

Now since y is any function of x ,

$$y^n D(y_1, y_2, \dots, y_n) \equiv \begin{vmatrix} (y, y) & (y, y)_1 & \dots & (y, y)_{n-1} \\ (y_2, y) & (y_2, y)_1 & \dots & (y_2, y)_{n-1} \\ \dots & \dots & \dots & \dots \\ (y_n, y) & (y_n, y)_1 & \dots & (y_n, y)_{n-1} \end{vmatrix}, \quad (5)$$

in which the subscript k of $(y_i, y)_k$ means the k th derivative of (y_i, y) . That is to say, the Wronskian of $y, y, y_2 y, \dots, y_n y$ is the product $y^n D(y_1, y_2, \dots, y_n)$, or is the Wronskian of y_1, y_2, \dots, y_n multiplied by y^n . This is made evident by noting that since

$$(y_i, y)_1 = y_{i1}, y + y_i y', (y_i, y)_2 = y_{i2} y + 2y_{i1} y' + y_i y'',$$

etc., where y', y'', \dots are the successive derivatives of y , the determinant in (5) becomes a sum of determinants, of which the first is the product $y^n D(y_1, y_2, \dots, y_n)$, and all the others vanish.

It is easily shown that

$$\frac{d D(y_1, y_2, \dots, y_n)}{dx} = \begin{vmatrix} y_1 & y_{11} & \dots & y_{1, n-2} & y_{1, n} \\ y_2 & y_{21} & \dots & y_{2, n-2} & y_{2, n} \\ \dots & \dots & \dots & \dots & \dots \\ y_n & y_{n1} & \dots & y_{n, n-2} & y_{n, n} \end{vmatrix}$$

for in the sum of determinants which make up the derivative, all vanish except the one expressed above.

If in equation (5) we put $y = \frac{1}{y_1}$, we see immediately that the first element in the first row becomes 1 and all other elements in the first row become 0. Hence, making this substitution and expanding by minors, the Wronskian reduces to

$$\begin{vmatrix} \left(\frac{y_2}{y_1}\right) & \left(\frac{y_2}{y_1}\right)_2 & \dots & \left(\frac{y_2}{y_1}\right)_{n-1} \\ \left(\frac{y_3}{y_1}\right) & \left(\frac{y_3}{y_1}\right)_2 & \dots & \left(\frac{y_3}{y_1}\right)_{n-1} \\ \dots & \dots & \dots & \dots \\ \left(\frac{y_n}{y_1}\right) & \left(\frac{y_n}{y_1}\right)_2 & \dots & \left(\frac{y_n}{y_1}\right)_{n-1} \end{vmatrix} = D \left[\left(\frac{y_2}{y_1}\right), \left(\frac{y_3}{y_1}\right), \dots, \left(\frac{y_n}{y_1}\right) \right].$$

But

$$\left(\frac{y_2}{y_1}\right) = \frac{D(y_1, y_2)}{y_1^2}, \quad \left(\frac{y_3}{y_1}\right) = \frac{D(y_1, y_3)}{y_1^2}, \dots, \left(\frac{y_n}{y_1}\right) = \frac{D(y_1, y_n)}{y_1^2}.$$

Now if we substitute

$$D(y_1, y_2) = z_2, \quad D(y_1, y_3) = z_3, \quad \dots, \quad D(y_1, y_n) = z_n$$

we get the equation

$$D(y_1, y_2, \dots, y_n) = \frac{1}{y_1^{n-2}} D(z_2, z_3, \dots, z_n). \quad (6)$$

It is easily shown that

$$\frac{\partial(y_1, y_2, \dots, y_m)}{\partial x} = \begin{vmatrix} y_1' & y_2' & \dots & y_m' \\ y_1'' & y_2'' & \dots & y_m'' \\ \dots & \dots & \dots & \dots \\ y_1^{(m)} & y_2^{(m)} & \dots & y_m^{(m)} \end{vmatrix}$$

for in the sum of determinants which make up the derivative, all vanish except the one expressed above. If in equation (5) we put $y = \frac{1}{y}$, we see immediately that the first element in the first row becomes 1 and all other elements in the first row become 0. Hence, making this substitution and expanding by minors, the formula reduces to

$$D \begin{bmatrix} \frac{y_1}{y} \\ \frac{y_2}{y} \\ \dots \\ \frac{y_m}{y} \end{bmatrix} = \begin{bmatrix} \left(\frac{y_1}{y}\right)' \left(\frac{y_2}{y}\right) \dots \left(\frac{y_m}{y}\right) \\ \left(\frac{y_1}{y}\right) \left(\frac{y_2}{y}\right)' \dots \left(\frac{y_m}{y}\right) \\ \dots \\ \left(\frac{y_1}{y}\right) \left(\frac{y_2}{y}\right) \dots \left(\frac{y_m}{y}\right)' \end{bmatrix}$$

But

$$\left(\frac{y_i}{y}\right)' = \frac{D(y_i, y)}{y^2} = \frac{D(y_1, y_2)}{y^2} \dots \frac{D(y_{i-1}, y_{i+1})}{y^2} \dots \frac{D(y_1, y_m)}{y^2}$$

Now if we substitute

$$D(y_1, y_2) = z_1, D(y_1, y_3) = z_2, \dots, D(y_1, y_m) = z_{m-1}$$

we get the equation

$$(5) \quad D(y_1, y_2, \dots, y_m) = \frac{1}{y^m} D(z_1, z_2, \dots, z_{m-1})$$

Now with the result of equation (6) we shall show that if the Wronskian of y_1, y_2, \dots, y_n vanishes, the functions are connected by a linear equation having constant coefficients.

Suppose that y_1 does not vanish, and since by hypothesis

$$D(y_1, y_2, \dots, y_n) = 0,$$

by equation (6) we also have

$$\frac{1}{y_1^{n-1}} D(z_2, z_3, \dots, z_n) = 0.$$

Therefore, $D(z_2, z_3, \dots, z_n)$ must vanish. Now assume that when this Wronskian vanishes, the $(n-1)$ functions z_2, z_3, \dots, z_n are connected by a linear relation, i. e.,

$$a_2 z_2 + a_3 z_3 + \dots + a_n z_n = 0. \quad (7)$$

Dividing equations (7) by y_1^n , and substituting the original values of z_2, z_3, \dots, z_n we have the following equation

$$a_2 \left\{ \frac{y_2}{y_1} \right\}_1 + a_3 \left\{ \frac{y_3}{y_1} \right\}_1 + \dots + a_n \left\{ \frac{y_n}{y_1} \right\}_1 = 0. \quad (8)$$

Integrating equation (8) we get

$$a_1 y_1 + a_2 y_2 + a_3 y_3 + \dots + a_n y_n = 0.$$

Therefore, assuming that if the Wronskian of $n-1$ functions vanishes the functions are connected by a linear relation, we have shown that when the Wronskian of n functions vanishes, the functions are connected by a linear relation. But we can show that the assumption is true for two functions, hence the theorem is true universally.

Let the function be $y_1 = f_1(x)$
 $y_2 = f_2(x).$

If the Wronskian vanishes

$$D(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_{1'} & y_{2'} \end{vmatrix} = 0,$$

or $y_1 y_{2'} - y_2 y_{1'} = 0.$

Integrating this gives

$$c_1 y_1 + c_2 y_2 = 0,$$

which proves the statement that when the Wronskian of two functions vanishes, they are connected by a linear relation.

Summary

After giving the fundamental properties of determinants with methods of evaluation, I have made a brief survey of the wide range of applications involving the use of determinants. Typical and important applications in the Theory of Equations, in the Calculus, in Trigonometry, and in Analytic Geometry have been given, but the list is by no means exhaustive. Many other interesting applications could well be included.

Bibliography

- Bôcher, Maxime Introduction to Higher Algebra
Macmillan Company, 1907.
- Doster, G. Éléments de la Théorie des Déterminants avec
Application à l'Algèbre, la Trigonométrie et
la Géométrie Analytique dans le Plan et dans
l'Espace.
Gauthier-Villars, second Edition--1883.
- Hanus, Paul H. An Elementary Treatise on the Theory of
Determinants.
Ginn and Company -- 1886
- Hart, William L. College Algebra
D. C. Heath and Company -- 1926
- Osgood, William F. Advanced Calculus
Macmillan Company -- 1925
- Weld, Laenas G. Determinants. No. 3 of the Mathematical
Monographs edited by Mansfield Merriman and
Robert S. Woodward. Fourth Edition--1906
Published by John Wiley & Sons.

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